

On backward stochastic differential equations driven by a family of Itô's processes.

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Abstract

We propose to study a new type of Backward stochastic differential equations, driven by a family of Itô's processes. We prove existence and uniqueness of the solution, and investigate stability and comparison theorem.

Keys Words: Backward stochastic differential equation; Family of Itô's processes; Dynamic sublinear expectation operator; m -stability; Hedging claims under model uncertainty.

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1 Introduction.

Originally motivated by questions arising in stochastic control theory, the theory of backward stochastic differential equations (BSDEs for short) has found important applications in fields as stochastic control, mathematical finance, Dynkin games and the second order PDE theory (see, for example, [6, 10, 9, 3, 4] and the references therein).

BSDEs have been introduced long time ago by J. B. Bismut [2] both as the equations for

the adjoint process in the stochastic version of Pontryagin maximum principle as well as the model behind the Black and Scholes formula for the pricing and hedging of options in mathematical finance. However the first published paper on nonlinear BSDEs appeared only in 1990, by Pardoux and Peng [9]. The classical BSDE consists of an equation of the form:

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s) ds - \int_t^T Z_s \cdot dW_s, \quad 0 \leq t \leq T. \quad (1.1)$$

driven by a d -dimensional Brownian motion W , with a deterministic terminal time $T > 0$, a generator $f : [0, T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$ and an \mathcal{F}_T -measurable terminal value ξ , where $(\mathcal{F}_t)_{t \leq T}$ is the natural filtration of $(W_t)_{t \leq T}$ augmented by the null sets. The solution of this equation, denoted by $eq(f, \xi)$, is a pair of adapted processes (Y, Z) with values in $\mathbb{R} \times \mathbb{R}^d$ and $Y_T = \xi$. The existence and uniqueness result of Pardoux and Peng assumes the uniform Lipschitz assumption on the generator f in y and z . So it is supposed that there exists a positive constant K such that:

$$|f(s, \omega, y, z) - f(s, \omega, y', z')| \leq K(|y - y'| + |z - z'|).$$

The proof of this Theorem is done in two steps. The first step consider the particular case where the generator f does not depend on the variables y and z . The process M defined for $t \in [0, T]$ by

$$M_t = \mathbb{E} \left(\xi + \int_0^T f(s) ds \middle| \mathcal{F}_t \right),$$

is a martingale, so by using the martingale representation theorem there exists an \mathbb{R}^d -valued integrable process Z such that $M_t = M_0 + \int_0^t Z_s \cdot dW_s$. We define then the process Y by

$$Y_t = \mathbb{E} \left(\xi + \int_t^T f(s) ds \middle| \mathcal{F}_t \right) = \xi + \int_t^T f(s) ds - \int_t^T Z_s \cdot dW_s.$$

The second step is based on a fixed point theorem: by introducing the Banach space $\mathcal{H}_{T, \beta}(\mathbb{R}^k)$ associated to the norm

$$\|X\|_{T, \beta} = \left(\mathbb{E} \int_0^T e^{\beta s} |X_s|^2 ds \right)^{1/2},$$

we define the mapping $\Phi : \mathcal{H}_{T, \beta}(\mathbb{R}) \times \mathcal{H}_{T, \beta}(\mathbb{R}^d) \rightarrow \mathcal{H}_{T, \beta}(\mathbb{R}) \times \mathcal{H}_{T, \beta}(\mathbb{R}^d)$ by $\Phi(y, z) = (Y, Z)$ where (Y, Z) is the solution of the BSDE with generator $f(s, y_s, z_s)$. Such solution exists from the first step. It is proved that Φ is a contraction for a specific value of β and then admits a unique fixed point.

Several papers extended these results by taking a more general driving martingale or by assuming weak assumptions on the generator f (Among others, see Antonelli [1], El Karoui and Huang [5], Ma, Protter and Yong [8], Pardoux and Peng [9, 10, 11], Peng [12, 13] and Essaky and Hassani [7]). For example the Pardoux-Peng result can be extended easily to the new equation $eq(f, \xi, S)$:

$$Y_t^S = \xi + \int_t^T f(s, Y_s^S, Z_s^S) ds - \int_t^T Z_s^S \cdot dS_s, \quad (1.2)$$

driven by an Itô process S of the form $dS_t = \mu_t^S dt + \sigma_t^S \cdot dW_t$, where μ^S and σ^S are respectively \mathbb{R}^d -valued and $\mathbb{R}^d \otimes \mathbb{R}^d$ -valued predictable processes. By supposing that the matrix-valued process σ^S is invertible and that the process $(\sigma^S)^{-1} \mu^S$ is uniformly bounded, we assure that S has a unique martingale measure denoted by \mathbb{Q}^S , equivalent to the physical probability \mathbb{P} . We remark that if (Y^S, Z^S) is the solution of the equation $eq(f, \xi, S)$, then $(Y^S, Z^S \sigma^S)$ is the solution of the equation $eq(f^S, \xi)$ where the generator f^S is defined by $f^S(t, y, z) = f(t, y, z(\sigma_t^S)^{-1}) - z(\sigma_t^S)^{-1} \mu_t^S$.

In this paper we consider a family D of Itô's processes instead of a single process S , such that each element of D admits a unique equivalent martingale measure. We propose to solve the equation $eq(f, \xi, D)$:

$$\hat{Y}_t = \xi + \int_t^T f(s, \hat{Y}_s, \hat{Z}_s) ds - \int_t^T \hat{Z}_s \cdot d\hat{S}_s, \quad (1.3)$$

for which the solution is a triplet $(\hat{Y}, \hat{Z}, \hat{S})$ satisfying the equation (1.3) and such that $\hat{S} \in D$ and the process $\int_0^\cdot \hat{Z}_s \cdot d\hat{S}_s$ is a \mathcal{E} -martingale with \mathcal{E} the dynamic sublinear expectation operator associated to the set of probability measures $\mathcal{Q} := \{\mathbb{Q}^S : S \in D\}$. We study existence and uniqueness of the solution, stability and comparison theorem.

This work is mainly motivated by pricing and hedging problems in Finance under model misspecification setting or the presence of some type of ambiguity.

2 Main Result

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \leq T}, P)$ be a stochastic basis on which is defined a d -dimensional Brownian motion $(W_t)_{t \leq T}$ such that $(\mathcal{F}_t)_{t \leq T}$ is the natural filtration of $(W_t)_{t \leq T}$ and \mathcal{F}_0 contains all P -null sets of \mathcal{F} . Note that $(\mathcal{F}_t)_{t \leq T}$ satisfies the usual conditions, *i.e.* it is right continuous and complete.

Let us now introduce the following notations :

- $\mathcal{L}_T^2(\mathbb{R})$ denotes the space of \mathcal{F}_T -measurable random variables ξ satisfying $\mathbb{E}|\xi|^2 < \infty$.
- $\mathcal{S}_T^2(\mathbb{R})$ is the space of predictable processes Y such that

$$\|Y\|^2 = \mathbb{E} \sup_{s \leq T} |Y_s|^2 < \infty.$$

- $\mathcal{H}_T^2(\mathbb{R}^d)$ is the space of predictable processes Z such that

$$\|Z\|^2 = \mathbb{E} \int_0^T |Z_s|^2 ds < \infty.$$

- \mathcal{P} denotes a set of predictable processes.
- \mathcal{S} is the set of \mathbb{R}^d -valued Itô's processes S of the form $dS = \mu^S dt + \sigma^S dW$.
- D is the set of process $S \in \mathcal{S}$ such that the process $\theta^S := (\sigma^S)^{-1} \mu^S$ belongs to \mathcal{P} and bounded.

We assume the following assumption **(H)** :

- (1) \mathcal{P} is predictably convex: for all $X^1, X^2 \in \mathcal{P}$ and a $\{0, 1\}$ -valued predictable process h , we have $X \in \mathcal{P}$ where $X_t = h_t X_t^1 + (1 - h_t) X_t^2$ for $t \in [0, T]$.
- (2) For any predictable process Z with values in \mathbb{R}^d , there exists some $S \in D$ such that
$$\text{ess} \inf_{S' \in D} (\theta_t^{S'} \cdot Z_t) = \theta_t^S \cdot Z_t, \text{ for all } t \in [0, T].$$
- (3) The terminal value $\xi \in \mathcal{L}_T^2(\mathbb{R})$, the generator f is uniformly K -Lipschitz with respect to y and z and $f(\cdot, 0, 0) \in \mathcal{H}_T^2(\mathbb{R})$.

Let us now introduce the definition of our BSDE driven by a family of Itô's processes.

Definition 2.1. A solution of the following BSDE eq(f, ξ, D)

$$\hat{Y}_t = \xi + \int_t^T f(s, \hat{Y}_s, \hat{Z}_s) ds - \int_t^T \hat{Z}_s \cdot d\hat{S}_s, \quad (2.1)$$

is a triplet $(\hat{Y}, \hat{Z}, \hat{S})$ satisfying equation (2.1) such that $\hat{S} \in D$ and the process $\int_0^\cdot \hat{Z}_s \cdot d\hat{S}_s$ is a \mathcal{E} -martingale with \mathcal{E} the dynamic sublinear expectation operator associated to the set of probability measures $\mathcal{Q} := \{\mathbb{Q}^S : S \in D\}$.

The main result of this paper is as follows.

Theorem 2.2. Under the assumption **(H)**, the equation eq(f, ξ, D) has a unique solution.

Before proving Theorem 2.2, we recall the definition of the m -stability property and state some intermediate results.

Definition 2.3. A family \mathcal{Q} of probability measures, all elements of which are equivalent to \mathbb{P} , is called multiplicativity stable (m -stable) if for all elements $\mathbb{Q}^1, \mathbb{Q}^2 \in \mathcal{Q}$ with density processes Λ^1, Λ^2 and for all stopping time $\tau \leq T$, it holds that $\Lambda_T := \Lambda_\tau^1 \Lambda_T^2 / \Lambda_\tau^2$ is the density of some $\mathbb{Q} \in \mathcal{Q}$.

Lemma 2.4. The set $\mathcal{Q} := \{\mathbb{Q}^S : S \in D\}$ is m -stable.

Proof. First each \mathbb{Q}^S is defined via its Radon-Nikodym density Λ_T^S , given by

$$\Lambda_T^S = \exp \left\{ \int_0^T \theta_s^S \cdot dW_s - \frac{1}{2} \int_0^T |\theta_s^S|^2 ds \right\}.$$

For $i = 1, 2$, let $S^i \in D$ with $\Lambda_T^{S^i}$ the density of \mathbb{Q}^{S^i} . Let a stopping time τ and define the probability measure \mathbb{Q} by its density $\Lambda_T = \Lambda_\tau^1 \Lambda_T^2 / \Lambda_\tau^2$ where $\Lambda_\tau^i = \mathbb{E}(\Lambda_T^i | \mathcal{F}_\tau)$. We shall prove that there exists some $S \in D$ such that $\mathbb{Q} = \mathbb{Q}^S$. For this we define the two processes μ and σ by $\mu_t = \mathbf{1}_{t < \tau} \mu_t^1 + \mathbf{1}_{t \geq \tau} \mu_t^2$ and $\sigma_t = \mathbf{1}_{t < \tau} \sigma_t^1 + \mathbf{1}_{t \geq \tau} \sigma_t^2$ for $t \in [0, T]$ where μ^i and σ^i are associated to S^i and define the process S by $dS = \mu dt + \sigma \cdot dW$. We verify easily from assumption **(H)**(1) by taking $h_t = \mathbf{1}_{\{t > \tau\}}$ that $\theta := \sigma^{-1} \mu \in \mathcal{P}$. So $S \in D$ and then $\mathbb{Q} = \mathbb{Q}^S$. ■

For the sake of clarity, we recall the following result which is a simplified version of Proposition 3.1 in [6].

Proposition 2.5. *For a family of standard parameters (f, ξ) and (f^α, ξ) , with α from an arbitrary index set, let (Y, Z) and (Y^α, Z^α) denote respectively the solution to the corresponding BSDEs $eq(f, \xi)$ and $eq(f^\alpha, \xi)$. If there exists a parameter $\bar{\alpha}$ such that*

$$f(t, Y_t, Z_t) = \text{ess inf}_{\alpha} f^\alpha(t, Y_t, Z_t) = f^{\bar{\alpha}}(t, Y_t, Z_t) \quad dP \times dt - a.e.,$$

then $Y_t = \text{ess inf}_{\alpha} Y_t^\alpha = Y_t^{\bar{\alpha}}$ holds for all $t \leq T$, P -a.s..

We suppose for the next two propositions that σ^S is the identity matrix for all $S \in D$. We define the dynamic sublinear expectation operator \mathcal{E} , associated to the set of probability measures $\mathcal{Q} = \{\mathbb{Q}^S : S \in D\}$, by $\mathcal{E}_t(X) = \text{ess sup}_{S \in D} \mathbb{E}^{\mathbb{Q}^S}(X | \mathcal{F}_t)$.

Proposition 2.6. *Let g be a square integrable adapted process and let (Y, Z) be the solution of the equation $eq(\hat{g}, \xi)$ where $\hat{g}(t, z) = \text{ess sup}_{S \in D} (g_t - \mu_t^S z)$. Then for $t \in [0, T]$*

$$Y_t = \mathcal{E}_t \left(\xi + \int_t^T g_s ds \right).$$

Proof. For $S \in D$, let (Y^S, Z^S) be the solution of the equation $eq(g, \xi, S)$. Then

$$Y_t^S = \mathbb{E}^{\mathbb{Q}^S} \left(\xi + \int_t^T g_s ds | \mathcal{F}_t \right).$$

We remark also that (Y^S, Z^S) is the solution of the equation $eq(g^S, \xi)$ where $g^S(t, z) = g_t - \mu_t^S z$ and then from Proposition 2.5 we deduce that

$$Y_t = \text{ess sup}_{S \in D} Y_t^S = \text{ess sup}_{S \in D} \mathbb{E}^{\mathbb{Q}^S} \left(\xi + \int_t^T g_s ds | \mathcal{F}_t \right).$$

Proposition 2.6 is then proved. ■

Proposition 2.7. *Under assumption **H**(2)-(3), there exists a unique solution (Y, Z) to the equation $eq(\hat{f}, \xi)$ where*

$$\hat{f}(t, y, z) = \text{ess sup}_{S \in D} (f(t, y, z) - \mu_t^S z).$$

Moreover, we have

$$Y_t = \mathcal{E}_t \left(\xi + \int_t^T f(s, Y_s, Z_s) ds \right).$$

Proof. Along the proof, C will denote a generic constant which may vary from line to line. The Proof is based on the Picard's approximation scheme. Let $(Y^0, Z^0) = (0, 0)$ and define (Y^{n+1}, Z^{n+1}) be the solution of the following BSDE:

$$Y_t^{n+1} = \xi + \int_t^T e^{s\beta} \sup_{S \in D} [f(s, Y_s^n, Z_s^n) - \mu_s^S Z_s^n] ds - \int_t^T Z_s^{n+1} \cdot dW_s. \quad (2.2)$$

Let $n, m \in \mathbb{N}$ and $\beta > 0$. Applying Itô's formula to $(Y_t^{n+1} - Y_t^{m+1})^2 e^{\beta t}$ we get

$$\begin{aligned} & (Y_t^{n+1} - Y_t^{m+1})^2 e^{\beta t} + \int_t^T e^{s\beta} |Z_s^{n+1} - Z_s^{m+1}|^2 ds + \int_t^T \beta e^{\beta s} (Y_s^{n+1} - Y_s^{m+1})^2 ds \\ &= 2 \int_t^T e^{s\beta} (Y_s^{n+1} - Y_s^{m+1}) (\hat{f}(s, Y_s^n, Z_s^n) - \hat{f}(s, Y_s^m, Z_s^m)) ds \\ & \quad - 2 \int_t^T e^{s\beta} (Y_s^{n+1} - Y_s^{m+1}) (Z_s^{n+1} - Z_s^{m+1}) \cdot dW_s, \end{aligned} \quad (2.3)$$

where $\hat{f}(t, Y, Z) = f(t, Y, Z) - \mu_t^{\hat{S}} Z$, for some $\hat{S} \in D$.

From assumptions **(H)**(3), it follows that for every $\varepsilon > 0$,

$$\begin{aligned} & 2(Y_s^{n+1} - Y_s^{m+1}) (\hat{f}(s, Y_s^n, Z_s^n) - \hat{f}(s, Y_s^m, Z_s^m)) \\ & \leq 2\varepsilon |Y_s^{n+1} - Y_s^{m+1}|^2 + \frac{K^2}{\varepsilon} |Y_s^n - Y_s^m|^2 + \frac{(K+C)^2}{\varepsilon} |Z_s^n - Z_s^m|^2, \end{aligned}$$

where we have used the fact that $\mu^{\hat{S}}$ is bounded by a positive constant C . By taking $\varepsilon = \frac{\beta}{2}$, it follows then that

$$\begin{aligned} & (Y_t^{n+1} - Y_t^{m+1})^2 e^{\beta t} + \int_t^T e^{s\beta} |Z_s^{n+1} - Z_s^{m+1}|^2 ds \\ & \leq \frac{2K^2}{\beta} \int_t^T e^{s\beta} |Y_s^n - Y_s^m|^2 ds + \frac{2(K+C)^2}{\beta} \int_t^T e^{s\beta} |Z_s^n - Z_s^m|^2 ds \\ & \quad - 2 \int_t^T e^{s\beta} (Y_s^{n+1} - Y_s^{m+1}) (Z_s^{n+1} - Z_s^{m+1}) \cdot dW_s. \end{aligned} \quad (2.4)$$

Using a localization procedure, we have

$$\begin{aligned} & \mathbb{E} \int_0^T e^{s\beta} |Z_s^{n+1} - Z_s^{m+1}|^2 ds \\ & \leq \frac{2(K+C)^2(T+1)}{\beta} \mathbb{E} \left[\sup_{s \leq T} e^{s\beta} |Y_s^n - Y_s^m|^2 + \int_0^T e^{s\beta} |Z_s^n - Z_s^m|^2 ds \right]. \end{aligned} \quad (2.5)$$

It follows from (2.4) and Davis-Burkholder-Gundy inequality that there exists a constant

$c > 0$, such that

$$\begin{aligned}
& \mathbb{E} \sup_{t \leq T} (Y_t^{n+1} - Y_t^{m+1})^2 e^{\beta t} \\
& \leq \frac{2(K+C)^2(T+1)}{\beta} \mathbb{E} \left[\sup_{s \leq T} e^{s\beta} |Y_s^n - Y_s^m|^2 + \int_0^T e^{s\beta} |Z_s^n - Z_s^m|^2 ds \right] \\
& \quad + c \mathbb{E} \left(\int_0^T (e^{s\beta})^2 |Y_s^{n+1} - Y_s^{m+1}|^2 |Z_s^{n+1} - Z_s^{m+1}|^2 ds \right)^{\frac{1}{2}} \\
& \leq \frac{2(K+C)^2(T+1)}{\beta} \mathbb{E} \left[\sup_{s \leq T} e^{s\beta} |Y_s^n - Y_s^m|^2 + \int_0^T e^{s\beta} |Z_s^n - Z_s^m|^2 ds \right] \\
& \quad + \frac{1}{2} \mathbb{E} \sup_{t \leq T} (Y_t^{n+1} - Y_t^{m+1})^2 e^{t\beta} + \frac{c^2}{2} \mathbb{E} \int_0^T e^{s\beta} |Z_s^{n+1} - Z_s^{m+1}|^2 ds.
\end{aligned}$$

By using inequality (2.5), we get

$$\begin{aligned}
& \mathbb{E} \sup_{t \leq T} (Y_t^{n+1} - Y_t^{m+1})^2 e^{\beta t} \\
& \leq \frac{2(K+C)^2(T+1)}{\beta} \left(\frac{c^2}{2} + 1 \right) \mathbb{E} \left[\sup_{s \leq T} e^{s\beta} |Y_s^n - Y_s^m|^2 + \int_0^T e^{s\beta} |Z_s^n - Z_s^m|^2 ds \right] \\
& \quad + \frac{1}{2} \mathbb{E} \sup_{t \leq T} (Y_t^{n+1} - Y_t^{m+1})^2 e^{t\beta},
\end{aligned}$$

and then

$$\begin{aligned}
& \mathbb{E} \sup_{t \leq T} (Y_t^{n+1} - Y_t^{m+1})^2 e^{\beta t} \\
& \leq \frac{4(K+C)^2(T+1)}{\beta} \left(\frac{c^2}{2} + 1 \right) \mathbb{E} \left[\sup_{s \leq T} e^{s\beta} |Y_s^n - Y_s^m|^2 + \int_0^T e^{s\beta} |Z_s^n - Z_s^m|^2 ds \right].
\end{aligned}$$

Coming back to equation (2.4), we have

$$\begin{aligned}
& \mathbb{E} \sup_{t \leq T} (Y_t^{n+1} - Y_t^{m+1})^2 e^{\beta t} + \mathbb{E} \int_0^T e^{s\beta} |Z_s^{n+1} - Z_s^{m+1}|^2 ds \\
& \leq \frac{4(K+C)^2}{\beta} (c^2 + 2)(T+1) \left(\mathbb{E} \sup_{t \leq T} (Y_t^n - Y_t^m)^2 e^{t\beta} + \mathbb{E} \int_0^T e^{s\beta} |Z_s^n - Z_s^m|^2 ds \right). \tag{2.6}
\end{aligned}$$

Taking $\beta \geq 16(K+C)^2(c^2+2)(T+1)$ and

$$\Gamma^{n+1, m+1} = \left(\mathbb{E} \sup_{t \leq T} (Y_t^{n+1} - Y_t^{m+1})^2 e^{t\beta} + \mathbb{E} \int_0^T e^{s\beta} |Z_s^{n+1} - Z_s^{m+1}|^2 ds \right)^{\frac{1}{2}},$$

it follows that, for every $n \geq m$

$$\Gamma^{n+1, m+1} \leq \frac{1}{2} \Gamma^{n, m} \leq \left(\frac{1}{2} \right)^m \Gamma^{n-m+1, 1}.$$

Using similar arguments as above and the fact that f is K -Lipschitz and $f(., 0, 0) \in \mathcal{H}_T^2(\mathbb{R})$, it is not difficult to prove that there exists a positive constant C such that

$$\Gamma^{n-m+1, 1} \leq C.$$

Therefore

$$\Gamma^{n+1,m+1} \leq C \left(\frac{1}{2} \right)^m,$$

and then

$$\lim_{n,m \rightarrow +\infty} \mathbb{E} \sup_{t \leq T} (Y_t^n - Y_t^m)^2 = 0, \quad \lim_{n,m \rightarrow +\infty} \mathbb{E} \int_0^T |Z_s^n - Z_s^m|^2 ds = 0.$$

Consequently, the sequence (Y^n, Z^n) converge to (Y, Z) in $\mathcal{S}_T^2(\mathbb{R}) \times \mathcal{H}_T^2(\mathbb{R}^d)$. Let (\bar{Y}, \bar{Z}) be the solution, which exists according to the previous proposition, of the following BSDE

$$\bar{Y}_t = \xi + \int_t^T \operatorname{ess\,sup}_{S \in D} [f(s, Y_s, Z_s) - \mu_s^S Z_s] ds - \int_t^T \bar{Z}_s \cdot dW_s.$$

It is not difficult to prove that there exists a constant $C > 0$ such that

$$\begin{aligned} & \mathbb{E} \sup_{t \leq T} (Y_t^{n+1} - \bar{Y}_t)^2 + \mathbb{E} \int_0^T |Z_s^{n+1} - \bar{Z}_s|^2 ds \\ & \leq C \left[\mathbb{E} \sup_{t \leq T} (Y_t^n - Y_t)^2 + \mathbb{E} \int_0^T |Z_s^n - Z_s|^2 ds \right]. \end{aligned}$$

Hence

$$\lim_{n \rightarrow +\infty} \left[\mathbb{E} \sup_{t \leq T} (Y_t^{n+1} - \bar{Y}_t)^2 + \mathbb{E} \int_0^T |Z_s^{n+1} - \bar{Z}_s|^2 ds \right] = 0.$$

It follows that

$$\mathbb{E} \sup_{s \leq T} |Y_s - \bar{Y}_s|^2 = 0, \text{ and } \mathbb{E} \int_0^T |Z_s - \bar{Z}_s|^2 ds = 0.$$

Therefore $Y = \bar{Y}$ and $Z = \bar{Z}$, and then (Y, Z) satisfies $eq(\xi, \hat{f})$.

Thanks to Proposition 2.6, it follows from Equation (2.2) that

$$Y_t^{n+1} = \mathcal{E}_t \left(\xi + \int_t^T f(s, Y_s^n, Z_s^n) ds \right). \quad (2.7)$$

By taking the limit in (2.7) and using the Fatou property of \mathcal{E} we obtain that

$$Y_t \geq \mathcal{E}_t \left(\xi + \int_t^T f(s, Y_s, Z_s) ds \right).$$

Since there exists some $\hat{S} \in D$ such that $\hat{f}(t, Y_t, Z_t) = f(t, Y_t, Z_t) - \mu_t^{\hat{S}} \cdot Z_t$. So (Y, Z, \hat{S}) is the solution of $eq(f, \xi, S)$, we take the expectation with respect to $\mathbb{Q}^{\hat{S}}$ in both parts of this equation and obtain that

$$Y_t = \mathbb{E}^{\mathbb{Q}^{\hat{S}}} \left(\xi + \int_t^T f(s, Y_s, Z_s) ds | \mathcal{F}_t \right) \leq \mathcal{E}_t \left(\xi + \int_t^T f(s, Y_s, Z_s) ds \right).$$

Therefore the second assertion is obtained. Proposition 2.7 is then proved. ■

Now we prove Theorem 2.2.

Proof of Theorem 2.2. Existence of a solution: Let (Y, Z) be the solution of the equation $eq(\hat{f}, \xi)$, where

$$\hat{f}(t, y, z) = \operatorname{ess\,sup}_{S \in D} (f(t, y, z) - \theta_t^S \cdot z),$$

and let $\hat{S} \in D$ such that $\hat{f}(t, Y, Z) = f(t, Y, Z) - \theta_t^{\hat{S}} \cdot Z$. By applying Proposition 2.7 to the family $\{\theta^S : S \in D\}$ instead of the family $\{\mu^S : S \in D\}$ we get that

$$Y_t = \mathcal{E}_t \left(\xi + \int_t^T f(s, Y_s, Z_s) ds \right).$$

But for $\hat{Z} = Z(\sigma^{\hat{S}})^{-1}$ we have that

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s) ds - \int_t^T \hat{Z}_s \cdot d\hat{S}_s$$

Then for all $t \in [0, T]$,

$$\mathcal{E}_t \left(\int_t^T \hat{Z}_s \cdot d\hat{S}_s \right) = 0.$$

Therefore (Y, \hat{Z}, \hat{S}) is a solution of the equation $eq(f, \xi, D)$.

Uniqueness of the solution: Let $(\hat{Y}^1, \hat{Z}^1, \hat{S}^1)$ and $(\hat{Y}^2, \hat{Z}^2, \hat{S}^2)$ two solutions of the equation $eq(f, \xi, D)$. By applying Itô formula to the semi-martingale $e^{\beta s}(\hat{Y}_s^1 - \hat{Y}_s^2)^2$ from $s = t$ to $s = T$ we obtain that

$$\begin{aligned} & e^{\beta t}(\delta \hat{Y}_t)^2 + \beta \int_t^T e^{\beta s}(\delta \hat{Y}_s)^2 ds + \int_t^T e^{\beta s} |\delta(\hat{Z}_s \sigma_s)|^2 ds \\ &= e^{\beta T}(\delta \hat{Y}_T)^2 + 2 \int_t^T e^{\beta s} \delta \hat{Y}_s \delta f(s) ds - \int_t^T e^{\beta s} \delta \hat{Y}_s d(\delta M)_s, \end{aligned} \quad (2.8)$$

with $\delta \hat{Y} = \hat{Y}^1 - \hat{Y}^2$, $\delta(\hat{Z} \sigma) = \hat{Z}^1 \sigma^{\hat{S}^1} - \hat{Z}^2 \sigma^{\hat{S}^2}$, $\delta f(s) = f(s, \hat{Y}^1, \hat{Z}^1 \sigma^{\hat{S}^1}) - f(s, \hat{Y}^2, \hat{Z}^2 \sigma^{\hat{S}^2})$, $\delta M = M^1 - M^2$ and $dM^i = \hat{Z}^i \cdot d\hat{S}^i$ for $i = 1, 2$. We need to show that the random variable $K := \int_t^T e^{\beta s} \delta \hat{Y}_s d\delta M_s$ has a positive expected value under a certain probability $\mathbb{Q} \in \mathcal{Q}$. Let define $\mu_t = \mathbf{1}_{(\delta \hat{Y}_t \geq 0)} \mu_t^1 + \mathbf{1}_{(\delta \hat{Y}_t < 0)} \mu_t^2$, $\sigma_t = \mathbf{1}_{(\delta \hat{Y}_t \geq 0)} \sigma_t^1 + \mathbf{1}_{(\delta \hat{Y}_t < 0)} \sigma_t^2$ and the process S by $dS = \mu dt + \sigma \cdot dW$. From assumption **(H)**(1) we have $S \in D$ and since M^1 and M^2 are \mathbb{Q}^S -super martingales, then

$$\eta_1 := \mathbb{E}^{\mathbb{Q}^S} \left(\int_t^T e^{\beta s} (\delta \hat{Y}_s)_- dM_s^1 \right) \leq 0,$$

and

$$\eta_2 := \mathbb{E}^{\mathbb{Q}^S} \left(\int_t^T e^{\beta s} (\delta \hat{Y}_s)_+ dM_s^2 \right) \leq 0,$$

where $(\delta\hat{Y}_s)_+$ and $(\delta\hat{Y}_s)_-$ are respectively the positive and the negative parts of $\delta\hat{Y}_s$. We have also that

$$\eta := \int_t^T e^{\beta s} (\delta\hat{Y}_s)_+ d\delta M_s^1 + \int_t^T e^{\beta s} (\delta\hat{Y}_s)_- d\delta M_s^2 = \int_t^T e^{\beta s} (\delta\hat{Y}_s) Z_s dS_s,$$

with $Z_s = \mathbf{1}_{\delta\hat{Y}_s \geq 0} \hat{Z}_s^1 + \mathbf{1}_{\delta\hat{Y}_s < 0} \hat{Z}_s^2$. Therefore $\eta_3 := \mathbb{E}^{\mathbb{Q}^S}(\eta) = 0$ and $\mathbb{E}^{\mathbb{Q}^S}(K) = -\eta_1 - \eta_2 + \eta_3 \geq 0$.

Now by taking the expectation in (2.8) with respect to $\mathbb{Q} = \mathbb{Q}^S$ and by following the same techniques as in Proposition 2.1 in [6] we obtain that $\delta\hat{Y} \equiv 0$ and $\delta(\hat{Z}\sigma) \equiv 0$.

■

Remark 2.8. It should be pointed out that our existence result hold true if we suppose that $\det(\sigma^S(\sigma^S)^{tr}) \neq 0, dP \times dt - a.e.$ for all $S \in \mathcal{S}$ and the set D is taken as follows $D = \{S \in \mathcal{S} : \theta^S := \sigma^S(\sigma^S)^{tr})^{-1} \mu^S \in \mathcal{P}, \text{ and bounded}\}$, where \mathcal{P} satisfying assumption **(H)**(1).

An immediate consequence of Theorem 2.2 concerns the generalization of the martingale representation of a square integrable random variable.

Corollary 2.9. For any $\xi \in \mathcal{L}_T^2(\mathbb{R})$, there exists a real number x_0 , a driver $\hat{S} \in D$ and a square integrable predictable \mathbb{R}^d -valued process \hat{Z} such that

$$\xi = x_0 + \int_0^T \hat{Z}_s \cdot d\hat{S}_s,$$

and the process $\left(\int_0^t \hat{Z}_s \cdot d\hat{S}_s\right)_{t \in [0, T]}$ is a \mathcal{E} -martingale, i.e for all $t < u$ we have

$$\mathcal{E}_t \left(\int_0^u \hat{Z}_s \cdot d\hat{S}_s \right) = \int_0^t \hat{Z}_s \cdot d\hat{S}_s.$$

Remark 2.10. The triplet $(\hat{Y}, \hat{Z}, \hat{S})$ is the unique solution of the equation $eq(f, \xi, D)$ if and only if $\hat{Y} = Y$, $\hat{Z} = Z(\sigma^{\hat{S}})^{-1}$ and $\text{ess inf}_{S \in D}(\theta^S \cdot Z) = \theta^{\hat{S}} \cdot Z$ where the pair (Y, Z) is the unique solution of the equation $eq(\hat{f}, \xi)$ with

$$\hat{f}(t, y, z) = \text{ess sup}_{S \in D} (f(t, y, z) - \theta_t^S z).$$

Thanks to the previous remark we obtain comparison theorem of solutions as a direct consequence of Theorem 2.2 in [6].

Theorem 2.11. Let $(\hat{Y}^i, \hat{Z}^i, \hat{S}^i)$ be the solution of the equation $eq(f^i, \xi^i, D)$ for $i = 1, 2$. We suppose that $\xi^1 \geq \xi^2$ a.s and that $\delta_2 f_t := f^1(t, \hat{Y}_t^2, \hat{Z}^2 \hat{\sigma}_t^2) - f^2(t, \hat{Y}_t^2, \hat{Z}^2 \hat{\sigma}_t^2) \geq 0$ a.s. Then for a.s any time t we have $\hat{Y}_t^1 \geq \hat{Y}_t^2$. Moreover if $\hat{Y}_t^1 = \hat{Y}_t^2$ on a \mathcal{F}_t -measurable set A , then $\hat{Y}_s^1 = \hat{Y}_s^2$ on $[t, T] \times A$.

Proof. We have $f^1(t, \hat{Y}_t^2, \hat{Z}^2 \hat{\sigma}_t^2) \geq f^2(t, \hat{Y}_t^2, \hat{Z}^2 \hat{\sigma}_t^2)$, then $\hat{f}^1(t, \hat{Y}_t^2, \hat{Z}^2 \hat{\sigma}_t^2) \geq \hat{f}^2(t, \hat{Y}_t^2, \hat{Z}^2 \hat{\sigma}_t^2)$. Thanks to Theorem 2.2 in [6] and Remark 2.10 we obtain the result.

■

Another consequence of remark 2.10 concerns the explicit solution of linear BSDE.

Corollary 2.12. *Let (α, γ) be a bounded $\mathbb{R} \times \mathbb{R}^d$ -valued predictable process, $\varphi \in \mathcal{H}_T^2(\mathbb{R})$ and $\xi \in \mathcal{L}_T^2(\mathbb{R})$. Then the linear BSDE $\text{eq}(\xi, f, D)$ with $f(s, y, z) = \varphi_s + y\alpha_s + z \cdot \gamma_s$,*

$$Y_t = \xi + \int_t^T (\varphi_s + Y_s \alpha_s + Z_s \cdot \gamma_s) ds - \int_t^T Z_s \cdot dS_s,$$

has a unique solution $(\hat{Y}, \hat{Z}, \hat{S})$ such that \hat{S} is solution of the minimization problem

$$\text{ess} \inf_{S \in D} (\theta^S \cdot Z) = \theta^{\hat{S}} \cdot Z,$$

for all predictable processes Z and \hat{Y} is given by

$$\hat{Y}_t = \Gamma_t^{-1} \mathbb{E} \left(\xi \Gamma_T + \int_t^T \varphi_s \Gamma_s ds \mid \mathcal{F}_t \right),$$

where Γ_s is the adjoint process defined for $s \geq 0$ by the forward linear SDE:

$$d\Gamma_s = \Gamma_s \left(\alpha_s ds + (\sigma_s^{\hat{S}})^{-1} (\gamma_s - \mu_s^{\hat{S}}) \cdot dW_s \right),$$

and $\Gamma_0 = 1$.

Next we illustrate previous results by an example of a geometric Brownian motion with volatility uncertainty.

Example 2.13. *We consider the case $d = 1$ and define the geometric Brownian motion S , solution of the equation*

$$dS_t = \mu S_t dt + \sigma^S S_t dW_t,$$

and define the family D as the set of processes S that satisfy $\sigma^S \in [\sigma^1, \sigma^2]$ where σ^1 and σ^2 are two positive real constants. So the equation $\text{eq}(f, \xi, D)$ has the unique solution $(\hat{Y}, \hat{Z}, \hat{S})$ given by

$$\hat{Y} = Y, \quad \hat{Z} = \frac{Z}{\sigma(Z)}, \quad d\hat{S} = \mu \hat{S} dt + \sigma(Z) \hat{S} dW,$$

with $\sigma(Z) = \sigma^1 \mathbf{1}_{(Z \geq 0)} + \sigma^2 \mathbf{1}_{(Z < 0)}$ and the pair (Y, Z) is the unique solution of the equation $\text{eq}(\hat{f}, \xi)$ where $\hat{f}(t, y, z) = f(t, y, z) - \frac{\mu}{\sigma(z)} z$.

3 Application to hedging claims under model uncertainty.

We consider a financial market, which is composed of a riskless asset and d risky assets. We suppose that the price of these $d + 1$ assets is modelled as follows: $S^0 \equiv 1$ and $S = (S^1, \dots, S^d)$ is solution of the stochastic differential equation:

$$dS_t = S_t (\mu_t dt + \sigma_t \cdot dW_t),$$

where μ and σ are respectively \mathbb{R}^d -valued and $\mathbb{R}^d \otimes \mathbb{R}^d$ -valued predictable processes. By supposing that the matrix-valued process σ is invertible and that the process $\sigma^{-1}\mu$ is uniformly bounded, we assure that S has a unique martingale measure denoted by \mathbb{Q}^S , equivalent to the physical probability \mathbb{P} , and therefore every contingent claim with payoff value H at maturity time T can be fully hedged, which means that there exists an \mathbb{R}^d -valued strategy ϕ and a price $\mathbb{E}^{\mathbb{Q}^S}(H)$ such that $H = \mathbb{E}^{\mathbb{Q}^S}(H) + \int_0^T \phi_s \cdot dS_s$. In the Markovian case and for an European type option contract $H = f(S_T)$ we can express the strategy ϕ as follows: we define the function $u(t, x) = \mathbb{E}(f(S_{T-t}) | S_0 = x)$ and found out that $H = u(0, S_0)$ and u is the solution of the partial differential equation:

$$\partial_t u + \mu \partial_x u + \frac{1}{2} \sigma \sigma^* \partial_x^2 u = 0,$$

and $u(T, x) = f(x)$. In a general setting we express the strategy ϕ in terms of the solution of a backward stochastic differential equation: $H = Y_T$ where (Y, Z) is the solution of BSDE:

$$Y_t = H - \int_t^T Z_s \cdot dS_s.$$

We may ask if the full hedge is still possible and what is the price if we suppose some uncertainty on the parameters μ and/or σ . More precisely we shall consider the situation where the vector valued parameter $\theta_t := \sigma_t^{-1} \mu_t$ varies in a random interval $[h_t, g_t]$ for $t \in [0, T]$. We denote $D = \{S \in \mathcal{S} : \theta \in [h, g] \text{ a.s.}\}$.

Theorem 3.1. *Let a contingent claim H with $\mathcal{E}(H) < \infty$. Then $H = \hat{Y}_T$ where $(\hat{Y}, \hat{Z}, \hat{S})$ is the solution of the BSDE eq(0, H, D):*

$$\hat{Y}_t = H - \int_t^T \hat{Z}_s \cdot d\hat{S}_s,$$

and the process \hat{S} is given by $d\hat{S} = \sigma(\theta^0 dt + dW)$ with $\theta^0 = (h^i \mathbf{1}_{(Z^i > 0)} + g^i \mathbf{1}_{(Z^i \leq 0)})_{i=1 \dots d}$ and (Y, Z) is the solution of the BSDE:

$$Y_t = H - \int_t^T (h_s \cdot Z_s^- + g_s \cdot Z_s^+) dt - \int_t^T Z_s \cdot dW,$$

with $Z^+ = ((Z^1)^+, \dots, (Z^d)^+)$ and $Z^- = ((Z^1)^-, \dots, (Z^d)^-)$, where z^+ and z^- denote respectively the positive and the negative parts of z .

Remark 3.2. *We refer to [6] among other references for more details and motivations on BSDE's and their applications in numerous domains.*

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